III (H) Paper-V

Schwartz's Theorem

Theorem:— Let (a,b) be a point of the domain of a function f such that

(i) f_x exists in a certain neighbourhood of (a,b).

(ii) f_{xy} is continuous at (a,b)

Then $f_{yx}(a,b)$ exists and is equal to $f_{xy}(a,b)$.

Proof:— The given condition imply that there exists a certain nbd. of (a,b) at every point (x,y) of which $f_x(x,y)$, $f_y(x,y)$ and $f_{xy}(x,y)$ all exist.

Let $(a+h, b+k)$ be any point of the nbd.

We write

$$F(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$$

$$g(y) = f(a+h, y) - f(a,y)$$

So that $F(h,k) = g(b+k) - g(b)$ ——— (1)

Since f_y exists in a nbd. of (a,b), the function g of one variable y is derivable in $[b, b+k]$ and therefore by applying the mean-value theorem to the expression on the right of (1), we have

$$F(h,k) = kg'(b+\theta k) \text{ where } 0<\theta<1$$

$$= k[f_y(a+h, b+\theta k) - f_y(a, b+\theta k)] \quad \text{——— (2)}$$

Again since f_{xy} exists in a nbd of (a,b), therefore by applying the mean-value theorem to the right of (2), we get

$$F(h,k) = k[h f_{xy}(a+\theta' h, b+\theta k)] \text{ where } 0<\theta'<1$$

$$\Rightarrow \frac{F(h,k)}{hk} = f_{xy}(a+\theta' h, b+\theta k)$$

$$\Rightarrow \frac{1}{k}\left[\frac{f(a+h, b+k) - f(a, b+k)}{h} - \frac{f(a+h, b) - f(a,b)}{h}\right]$$

$$= f_{xy}(a+\theta' h, b+\theta k).$$

Since f_x exists in a nbd. of (a,b), therefore on taking the limit as $h \to 0$, we get

$$\frac{1}{k}[f_x(a, b+k) - f_x(a,b)] = \lim_{h\to 0} f_{xy}(a+\theta' h, b+\theta k)$$

Now, let $k \to 0$. Then

$$\lim_{k\to 0} \frac{f_x(a, b+k) - f_x(a,b)}{k}$$

$$= \lim_{k \to 0} \lim_{h \to 0} f_{xy}(a+\theta' h,\ b+\theta k)$$

$$\Rightarrow f_{yx}(a,b) = f_{xy}(a,b),$$

Since f_{xy} is continuous at (a,b).

This proves the theorem.